Rank Preserving Maps on CSL Algebras

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ABSTRACT. We give a description of a weakly continuous rank preserving map on a reflexive algebra on complex Hilbert space with commutative completely distributive subspace lattice. We show that the implementation of a rank preserving map can be described by the combination of two different types of maps. We also show that a rank preserving map can be implemented by only one type if the corresponding lattice is irreducible. We present some examples of both types of rank preserving map.

1. PRELIMINARIES

It has been of special interest to study the linear maps between two nonself-adjoint operator algebras over the last several decades. These maps include isomorphisms, isometries and rank preserving maps etc. In [**11**], Ringrose proved that an isomorphism between two nest algebras must be spatially implemented by making effective use of rank one operators. In [**3**], Gilfeather and Moore showed that an isomorphism acting between two CSL algebras with completely distributive lattices need not be spatial or quasi-spatially implemented. But they also showed that if an isomorphism preserves the rank of all finite rank operators, then it must be quasi-spatially implemented. In [**9**], Panaia has shown that if $\mathfrak{A}_i = \mathrm{Alg}\, \mathfrak{L}_i$ where the $\mathfrak{L}_i$ are finite distributive subspaces lattices, then every rank preserving algebraic isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ is quasi-spatial. The characterization of the rank preserving maps on nest algebras was done by the Chinese mathematicians, Shu-Yun Wei and Sheng-Zhao Hou [**12**]. Part of their results showed that some rank preserving maps Φ on a nest algebra can be described as $\Phi(T) = ATB^*$ for some $A, B \in B(\mathcal{H})$. Use of rank one operators was an integral part of their proof. In this paper, we shall describe the rank preserving maps on completely distributive commutative subspace lattice algebras (CDCSL algebras).

In this paper we show that a rank preserving map Φ on a completely distributive algebra, CDC algebra, with certain condition can be described as $\Phi(A) = UAV^*$ for some densely defined linear transformations U and V.

Let $\mathcal{H}$ be a complex separable Hilbert space. A *subspace lattice* $\mathfrak{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$, containing 0 and I. If $\mathfrak{L}$ is a subspace lattice, $\mathrm{Alg}\, \mathfrak{L}$ denotes the algebra of all bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathfrak{L}$. $\mathrm{Alg}\, \mathfrak{L}$ is a weakly closed subalgebra of $B(\mathcal{H})$, the algebra

of all bounded operators on $\mathcal{H}$. Dually, if $\mathfrak{A}$ is a subalgebra of $B(\mathcal{H})$, then $\operatorname{Lat}\mathfrak{A}$ is the lattice of all projections invariant for each operator in $\mathfrak{A}$. An algebra $\mathfrak{A}$ is reflexive if $\mathfrak{A} = \operatorname{Alg}\operatorname{Lat}\mathfrak{A}$ and a lattice $\mathfrak{L}$ is reflexive if $\mathfrak{L} = \operatorname{Lat}\operatorname{Alg}\mathfrak{L}$. A lattice is a *commutative subspace lattice*, or CSL, if each pair of projections in $\mathfrak{L}$ commute; $\operatorname{Alg}\mathfrak{L}$ is then called a CSL algebra. All lattices in this paper will be commutative.

In [**1**], Arveson showed that every commutative subspace lattice is reflexive. A totally ordered (and hence commutative) subspace lattice is a nest and the associated algebra is a nest algebra. We use the convention that a subspace is identified with the orthogonal projection onto the subspace. Thus $E \subseteq F$ is the same as $E \leq F$.

We have two operations between subspaces: meet($\wedge$) and join($\vee$).

Given any family $\{E_\alpha\}_{\alpha \in I}$ of subspaces of a Hilbert space $\mathcal{H}$, $\wedge_{\alpha \in I}E_\alpha$ denotes the greatest subspace contained in each E_α and $\vee_{\alpha \in I}E_\alpha$ denotes the smallest subspace containing each E_α.

We now restrict our attention to a special type of lattice: completely distributive lattice. One of the advantages of completely distributive lattices over other lattices is the abundance of the rank one operators in the associated algebra; these operators will be building blocks in the subsequent discussion. A *completely distributive lattice* is a complete lattice which satisfies distributive laws expressed for families of arbitrary cardinality. To be more precise, let $\mathfrak{L}$ be a complete lattice and I be an arbitrary index set. For each $\alpha \in I$, let J_α also be an arbitrary index set, and for each $\beta \in J_\alpha$, let $E_{\beta\alpha}$ denote an element of $\mathfrak{L}$. Π will denote the Cartesian product of all the J_α, i.e. the collection of all choice functions $\varphi : I \to \cup_{\alpha \in I}J_\alpha$ satisfying $\varphi(\alpha) \in J_\alpha$, for all α. $\mathfrak{L}$ is completely distributive if it satisfies the following two identities.

(1) $\wedge_{\alpha \in I}(\vee_{\beta \in J_\alpha}E_{\beta\alpha}) = \vee_{\varphi \in \Pi}(\wedge_{\alpha \in I}E_{\varphi(\alpha)\alpha})$
(2) $\vee_{\alpha \in I}(\wedge_{\beta \in J_\alpha}E_{\beta\alpha}) = \wedge_{\varphi \in \Pi}(\vee_{\alpha \in I}E_{\varphi(\alpha)\alpha})$

If $\mathfrak{L}$ is completely distributive and commutative, we will call $\operatorname{Alg}\mathfrak{L}$ a CDC algebra. For further discussion about the characteristics of CDC algebra, we need to introduce the notion of rank one operators. We will let $x \otimes y^*$ denote the rank one operator defined on $\mathcal{H}$ by $(x \otimes y^*)(f) = \langle f, y \rangle x$.

There are plenty of rank one operators and finite rank operators in a nest algebra. Moreover, these operators have played a central role in the theory of nest algebra. However, there are many examples of commutative subspace lattices whose corresponding algebras do not contain any rank one operators [**6**]. But as we discussed earlier, the complete distributivity of a lattice guarantees a reasonable supply of rank one operators. Furthermore, one of the surprising characteristics of the CDC algebra is stated in the following lemma. This lemma is due to Laurie and Longstaff [**7**].

LEMMA 1.1. *Let $\mathfrak{L}$ be a commutative completely distributive subspace lattice on a separable Hilbert space $\mathcal{H}$. Let $R_\mathfrak{L}$ denote the linear span of the rank one operators in $\operatorname{Alg}\mathfrak{L}$. Then $R_\mathfrak{L}$ is dense in $\operatorname{Alg}\mathfrak{L}$ in any of the weak, strong, ultraweak or ultrastrong topologies.*

It is an interesting question to ask when $x \otimes y^*$ belongs to $\operatorname{Alg}\mathfrak{L}$. The following lemma, due to Longstaff [**8**], is the answer to this question, and the use of this lemma is essential.

LEMMA 1.2. *The operator $x \otimes f^*$ belongs to $\text{Alg}\,\mathfrak{L}$ if and only if there is a projection $E \in \mathfrak{L}$ such that $x \in E$ and $f \in E_-^{\perp}$.*

Let $\mathfrak{L}$ be a subspace lattice. For $M \in \mathfrak{L}$, define M_- by $M_- = \vee\{N | M \neq N, N \in \mathfrak{L}\}$. We end this section with a couple of lemmas that will be used repeatedly.

LEMMA 1.3. *Let $\mathfrak{L}$ be a commutative completely distributive lattice. Then*

$$\vee\{N | N \in \mathfrak{L}, N_- \neq I\} = I$$

and

$$\vee\{N_-^{\perp} | N \in \mathfrak{L}, N \neq 0\} = I$$

LEMMA 1.4. *Let $\mathfrak{L}$ be a subspace lattice and $\{E_k\}_{k=1}^n$ be a finite subset of $\mathfrak{L}$. Then $(\vee_{k=1}^n E_k)_- = \vee_{k=1}^n (E_k)_-$ and $(\wedge_{k=1}^n E_k)_- \leq \wedge_{k=1}^n (E_k)_-$*

PROOF. It is sufficient to prove that the statements are true in case $n = 2$. Let E and F denote two elements in $\mathfrak{L}$. Since $E \leq E \vee F$ and $F \leq E \vee F$, we have $E_- \leq (E \vee F)_-$ and $F_- \leq (E \vee F)_-$. These imply that

$$E_- \vee F_- \leq (E \vee F)_-.$$

On the other hand, if $K \in \{G \in \mathfrak{L} \,|\, G \ngeq E \vee F\}$, then $K \in \{G \in \mathfrak{L} \,|\, G \ngeq E\}$ or $K \in \{G \in \mathfrak{L} \,|\, G \ngeq F\}$, so $K \leq E_-$ or $K \leq F_-$. These imply that $K \leq E_- \vee F_-$. Hence $(E \vee F)_- \leq E_- \vee F_-$. This proves (1).

Let E and F be two elements in $\mathfrak{L}$. Since $E \wedge F \leq E$ and $E \wedge F \leq F$, $(E \wedge F)_- \leq E_-$ and $(E \wedge F)_- \leq F_-$. Therefore $(E \wedge F)_- \leq E_- \wedge F_-$. The results follow. □

Now let's discuss two important types of linear maps on $\text{Alg}\,\mathfrak{L}$.

DEFINITION 1.5. *Let $\mathfrak{L}$ be a commutative subspace lattice.*

(1) A linear map $\Phi : \text{Alg}\,\mathfrak{L} \to \text{Alg}\,\mathfrak{L}$ is called an *isomorphism* if Φ is a bijection and a multiplication preserving map.

(2) A linear map $\Phi : \text{Alg}\,\mathfrak{L} \to \text{Alg}\,\mathfrak{L}$ is called a *rank preserving map* if $rank(\Phi(A)) = rank(A)$ for each finite rank operator A in $\text{Alg}\,\mathfrak{L}$.

The automatic norm continuity of an isomorphism is proved by Gilfeather and Moore in [3]. In order to discuss more results about isomorphisms, we need the following terminology. An isomorphism Φ is said to be *spatially implemented* if there is a bounded invertible operator T on $\mathcal{H}$ so that $\Phi(A) = TAT^{-1}$ for all $A \in \text{Alg}(\mathfrak{L})$. An isomorphism is said to be *quasi-spatially implemented* if there is a one-to-one operator with dense domain $\mathcal{D}$ so that $\Phi(A)Tf = TAf$ for all $A \in \text{Alg}\,\mathcal{L}$ and for all $f \in \mathcal{D}$.

The following theorem characterizes isomorphisms of CSL algebras. The theorem is due to Gilfeather and Moore [3].

THEOREM 1.6. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $\mathfrak{L}_1$ be completely distributive. Let $\rho : \text{Alg}\,\mathfrak{L}_1 \to \text{Alg}\,\mathfrak{L}_2$ be an algebraic isomorphism. The followings are equivalent.*

(1) *ρ is quasi-spatially implemented by a closed, injective linear transformation $T : H_1 \to H_2$ whose range and domain are dense.*

(2) *ρ is a rank-preserving map.*

The weak continuity of a rank preserving map Φ will be used in the proof of the main Theorem. However, we set the assumption that *each rank preserving map Φ is weakly continuous* from this early stage to avoid any misunderstanding.

2. RANK PRESERVING LINEAR MAPS

We begin with a lemma which will play a key role in our result. For notational convenience, we will let $a \sim b$ represent the linear dependency of two nonzero vectors a, b in $\mathcal{H}$. It can be easily verified that $a \sim b$ is an equivalence relation.

Suppose $\mathfrak{L}$ is a commutative subspace lattice and Φ is a rank preserving map from $\mathrm{Alg}\,\mathfrak{L}$ to $\mathrm{Alg}\,\mathfrak{L}$. Let N be an element in $\mathfrak{L}$ with $\dim N \geq 2$, $\dim N_-^\perp \geq 2$. Let x, y be any two linearly independent vectors in N and f, g be any two linearly independent vectors in $N_-^\perp$. Then we can form four different rank one operators in $\mathrm{Alg}\,\mathfrak{L}$ as follows: $x \otimes f^*, y \otimes f^*, x \otimes g^*, y \otimes g^*$.

Since Φ is a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$, we can consider another four rank one operators in $\mathrm{Alg}\,\mathfrak{L}$ which are images of each of the four rank one operators and write them as follows.

$$\Phi(x \otimes f^*) = u \otimes v^* \qquad\qquad \cdots\cdots (A)$$
$$\Phi(y \otimes f^*) = p \otimes q^* \qquad\qquad \cdots\cdots (B)$$
$$\Phi(x \otimes g^*) = w \otimes z^* \qquad\qquad \cdots\cdots (C)$$
$$\Phi(y \otimes g^*) = r \otimes s^* \qquad\qquad \cdots\cdots (D)$$

LEMMA 2.1. *(Four Vectors Lemma) Let $\mathfrak{L}$ be a commutative subspace lattice and Φ be a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$. Let N be a subspace in $\mathfrak{L}$ with $\dim N \geq 2$, $\dim N_-^\perp \geq 2$. If we consider the eight rank one operators described above, then either (1) or (2) holds for all $x, y \in N$ and $f, g \in N_-^\perp$.*

(1) $u \sim w, v \sim q, p \sim r$ *and* $z \sim s$
(2) $v \sim z, u \sim p, w \sim r$ *and* $q \sim r$

PROOF. Let's consider the following six equalities which are obtained by adding two of $(A), (B), (C)$ and (D).

$(A) + (C) : \Phi(x \otimes (f + g)^*) = u \otimes v^* + w \otimes z^*$
$(A) + (B) : \Phi((x + y) \otimes f^*) = u \otimes v^* + p \otimes q^*$
$(B) + (D) : \Phi(y \otimes (f + g)^*) = p \otimes q^* + r \otimes s^*$
$(C) + (D) : \Phi((x + y) \otimes g^*) = w \otimes z^* + r \otimes s^*$
$(A) + (D) : \Phi(x \otimes f^* + y \otimes g^*) = u \otimes v^* + r \otimes s^*$
$(B) + (C) : \Phi(x \otimes g^* + y \otimes f^*) = p \otimes q^* + w \otimes z^*$

Since $x \otimes (f + g)^*$ is a rank one operator and Φ is a rank preserving map, $u \otimes v^* + w \otimes z^*$ is also a rank one operator. This fact implies either $u \sim w$ or $v \sim z$. Similarly we can argue that either $u \sim p$ or $v \sim q$ is true by the equality $(A) + (B)$, either $p \sim r$ or $q \sim s$ is true by the equality $(B) + (D)$ and $w \sim r$ or $z \sim s$ is true by the fourth equality $(C) + (D)$. Apart from the first four equalities, the map Φ maps a rank two operator into a rank two operator in the equality $(A) + (D)$ and $(B) + (C)$, which gives the following results

$$u \nsim r, v \nsim s, p \nsim w \text{ and } q \nsim z.$$

Case 1: Now let's suppose $u \sim w$ is true. Then it is easily deduced that $v \sim q$. Otherwise $u \sim p$ is true implying $w \sim p$, which contradicts the relation $w \nsim p$. Similarly, we can show that $p \sim r, z \sim s$ are true, so the result follows.
Case 2: Suppose $v \sim z$. The proof of this case is almost identical to the proof of Case 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 2.2. If we assume that *Case 1* holds, then $u \nsim p$.

If we consider the case $\dim N = 1$ and $\dim N_-^\perp \geq 2$, we have the following two equalities.

$$\Phi(x \otimes f^*) = u \otimes v^* \qquad\qquad \cdots\cdots(E)$$
$$\Phi(x \otimes g^*) = p \otimes q^* \qquad\qquad \cdots\cdots(F)$$

where $x \in N$ and f, g are two linearly independent vectors in $N_-^\perp$. The following lemma is analogous to Lemma 2.1 for this case.

LEMMA 2.3. *Let $\mathfrak{L}$ be a commutative subspace lattice and Φ be a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$. Let N be a subspace in $\mathrm{Alg}\,\mathfrak{L}$ with $\dim N = 1$ and $\dim N_-^\perp \geq 2$. If we consider the two equalities described above, then either (1) or (2) holds for all $x \in N$ and $f, g \in N_-^\perp$.*

(1) $u \sim p$, $v \nsim q$
(2) $u \nsim p$, $v \sim q$

PROOF. By adding the two equalities, the following equality is obtained.
$(E) + (F) : \Phi(x \otimes (f + g)^*) = u \otimes v^* + p \otimes q^*$
From the fact that Φ maps a rank one operator into a rank one operator, it can be easily deduced that either $u \sim p$ or $v \sim q$ is true. Suppose $u \sim p$ and $v \sim q$. Then there exist $\lambda, \gamma \in \mathbb{C}$ such that $\lambda u \otimes v^* + \gamma p \otimes q^* = 0$. Therefore $\Phi(x \otimes (\overline{\lambda}f + \overline{\gamma}g)^*) = \lambda u \otimes v^* + \gamma p \otimes q^* = 0$ which is a contradiction that Φ maps a rank one operator into a rank zero operator, so the result follows. □

The next lemma states a different version of Lemma 2.3 for the case $\dim N \geq 2$ and $\dim N_-^\perp = 1$. The proof of this lemma is almost same as the proof of Lemma 2.3.

LEMMA 2.4. *Let $\mathfrak{L}$ be a commutative subspace lattice and Φ be a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$. Let N be a subspace in $\mathrm{Alg}\,\mathfrak{L}$ with $\dim N \geq 2$ and $\dim N_-^\perp = 1$. If we consider the following two equalities,*

$$\Phi(x \otimes f^*) = u \otimes v^* \qquad\qquad \cdots\cdots(G)$$
$$\Phi(y \otimes f^*) = p \otimes q^* \qquad\qquad \cdots\cdots(H)$$

where x, y are two linearly independent vectors in N and $f \in N_-^\perp$, then one of the following holds.

(1) $u \sim p$, $v \nsim q$
(2) $u \nsim q$, $v \sim q$

With these three lemmas in hand, we now can show the following important result.

LEMMA 2.5. *Let $\mathfrak{L}$ be a commutative subspace lattice and Φ be a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$. Let N be a nonzero element in $\mathfrak{L}$ with $N_- \neq I$. Then at least one of the following holds.*

(1) *There exist a linear map U from N to $\mathcal{H}$ and a linear map V from $N_-^\perp$ to $\mathcal{H}$ such that $\Phi(x \otimes f^*) = U(x) \otimes V(f)^*$ for all $x \in N$ and $f \in N_-^\perp$.*
(2) *There exist a conjugate linear map U from $N_-^\perp$ to $\mathcal{H}$ and a conjugate linear map V from N to $\mathcal{H}$ such that $\Phi(x \otimes f^*) = U(f) \otimes V(x)^*$ for all $x \in N$ and $f \in N_-^\perp$.*

PROOF. Suppose (1) holds for all $x, y \in N$ and $f, g \in N_-^\perp$ in Lemma 2.1, Lemma 2.3 and Lemma 2.4.

<u>Case 1:</u> Suppose that $\dim N = 1$ and $\dim N_-^\perp = 1$. Fix $x_1 \in N, f_1 \in N_-^\perp$. Set $\Phi(x_1 \otimes f_1^*) = u_1 \otimes v_1^*$. For any $x \in N, f \in N_-^\perp$, there exist $\lambda, \gamma \in \mathbb{C}$ such that $x = \lambda x_1, f = \gamma f_1$. If we define $U : N \to \mathcal{H}$ and $V : N_-^\perp \to \mathcal{H}$ by $U(x) = \lambda u_1$ and $V(f) = \gamma v_1$, then these maps are clearly well defined linear maps. However, if we define $U : N \to \mathcal{H}$ and $V : N_-^\perp \to \mathcal{H}$ by $U(x) = \lambda v_1$ and $V(f) = \gamma u_1$, these maps are also well defined linear maps. Therefore both conclusions (1) and (2) hold in this special case.

<u>Case 2:</u> Suppose that $\dim N = 1$ and $\dim N_-^\perp \geq 2$. Fix $x_1 \in N$. It follows from Lemma 2.3 (1) that it is possible to to define a map V on $N_-^\perp$ by $\Phi(x_1 \otimes f^*) = u_1 \otimes V(f)$. For any $f, g \in N_-^\perp, t \in \mathbb{C}$, since Φ is linear,

$$\Phi(x_1 \otimes (f + tg)^*) = \Phi(x_1 \otimes f^*) + \Phi(x_1 \otimes (tg)^*)$$
$$= u_1 \otimes V(f)^* + \bar{t} u_1 \otimes V(g)^*.$$

From the definition of V, $\Phi(x_1 \otimes (f + tg)^*) = u_1 \otimes V(f + tg)^*$. By comparing the two equalities, we see the map $V : N_-^\perp \to \mathcal{H}$ is a linear map on $N_-^\perp$.

For any $x \in N$, there exists $\lambda \in \mathbb{C}$ such that $x = \lambda x_1$. Now we can define a linear map $U : N \to \mathcal{H}$ by $U(x) = \lambda u_1$. Therefore the conclusion (1) holds with this assumption. Likewise, the conclusion (2) follows if we assume (2) in Lemma 2.3 is true.

<u>Case 3:</u> Suppose that $\dim N \geq 2$ and $\dim N_-^\perp = 1$. The proof of this case is almost identical to the proof of case 2.

<u>Case 4:</u> Let's fix $x_1 \in N$. By Lemma 2.1 (1), we can define a map V from $N_-^\perp$ to $\mathcal{H}$ by $\Phi(x_1 \otimes f^*) = u_1 \otimes V(f)^*$. For any $f, g \in N_-^\perp, t \in \mathbb{C}$, since Φ is linear,

$$\Phi(x_1 \otimes (f + tg)^*) = \Phi(x_1 \otimes f^*) + \Phi(x_1 \otimes (tg)^*)$$
$$= u_1 \otimes V(f)^* + \bar{t} u_1 \otimes V(g)^*.$$

From the definition of V, $\Phi(x_1 \otimes (f + tg)^*) = u_1 \otimes V(f + tg)^*$. By comparing the two equalities, we see the map V is a linear map on $N_-^\perp$. Likewise, we can define a linear map U from N to $\mathcal{H}$ by $\Phi(x \otimes f_1^*) = U(x) \otimes v_1^*$ for some fixed $f_1, v_1 \in N_-^\perp$. By considering

$$\Phi(x_1 \otimes f_1^*) = U(x_1) \otimes v_1^*$$
$$= u_1 \otimes V(f_1)^*$$
$$= u_1 \otimes v_1^*,$$

we can make an observation that $U(x_1) = u_1$ and $V(f_1) = v_1$. Now for any $x \in N, f \in N_-^\perp$, we may write $\Phi(x \otimes f^*)$ as follows.

$$\Phi(x \otimes f^*) = \alpha(x, f) U(x) \otimes V(f)^*$$

where $\alpha(x, f)$ is a complex valued function.

We claim that $\alpha(x, f) = 1$ for all x and f. Note that $\Phi(x_1 \otimes f^*) = \alpha(x_1, f) u_1 \otimes V(f)^* = u_1 \otimes V(f)^*$, so $\alpha(x_1, f) = 1$ for any $f \in N_-^\perp$. Similarly, we observe that

$\alpha(x, f_1) = 1$ for any $x \in N$. Consider

$$\Phi((x + sx') \otimes f^*) = \alpha(x + sx', f)(U(x + sx') \otimes V(f)^*)$$
$$= \alpha(x + sx', f)[(U(x) + sU(x')) \otimes V(f)^*]$$
$$= \alpha(x + sx', f)U(x) \otimes V(f)^* + s\alpha(x + sx', f)U(x') \otimes V(f)^*$$

where x and x' are linearly independent vectors in N.

On the other hand,

$$\phi((x + sx') \otimes f^*) = \Phi(x \otimes f^*) + s\Phi(x' \otimes f^*)$$
$$= \alpha(x, f)U(x) \otimes V(f)^* + s\alpha(x', f)U(x') \otimes V(f)^*.$$

We get the following equality by comparing the two equalities above:

$$\alpha(x + sx', f)U(x) + s\alpha(x + sx', f)U(x') = \alpha(x, f)U(x) + s\alpha(x', f)U(x')$$

for all $s \in \mathbb{C}$. Since $U(x)$ and $U(x')$ are linearly independent, $\alpha(x, f) = \alpha(x + sx', f) = \alpha(x', f)$. Hence, α is independent of x. Likewise we can show that α is independent of f. Therefore $\alpha \equiv 1$. The result follows. $\square$

What Lemma 2.5 says is that whether a given subspace N satisfies (1) or (2) in the lemma is a characteristic of the subspace which is associated with the rank preserving map Φ.

DEFINITION 2.6. Let $\mathfrak{L}$ be a commutative subspace lattice and Φ be a rank preserving map on $\text{Alg}\,\mathfrak{L}$. Let N be a nonzero subspace in $\text{Alg}\,\mathfrak{L}$. The subspace N is called *consistent with respect to* Φ if Φ and N satisfy (1) in Lemma 2.5. The subspace N is called *twisted with respect to* Φ if Φ and N satisfy (2) in Lemma 2.5.

In a nest algebra, it is impossible that both N and $N^\perp$ are in a nest $\mathcal{N}$ (a totally ordered lattice) if $N \neq 0$ and $N \neq I$. But a commutative subspace lattice $\mathfrak{L}$ can have N and $N^\perp$ both in it with the assumption $N \neq 0$ and $N \neq I$. If there exists a subspace $N \in \mathfrak{L}$ with $N_- \neq I$ such that $\dim N = 1$ and $N^\perp \in \mathfrak{L}$, then $N_- = \vee\{E \in \mathfrak{L} \,|\, E \not\geq N\} \geq N^\perp$ since $N^\perp \in \mathfrak{L}$ and $N^\perp \not\geq N$. Hence $N_-^\perp \leq N$. Since $N_- \neq I$, $N_-^\perp = N$. $\underline{\text{Case 1}}$ in the proof of Lemma 2.5 shows that such N is both consistent and twisted with respect to Φ.

DEFINITION 2.7. A nonzero subspace N in $\mathfrak{L}$ with $\dim N = 1$ and $N^\perp \in \mathfrak{L}$ is called *isolated*.

LEMMA 2.8. *Let Φ be a rank preserving map on $\text{Alg}\,\mathfrak{L}$ and N be a subspace in $\mathfrak{L}$. Then N is isolated if and only if N is both consistent and twisted with respect to Φ.*

PROOF. It suffices to prove sufficiency. Suppose N is both consistent and twisted with respect to Φ. Then there exist linear maps U_1, V_2 defined on N, and U_2, V_1 defined on $N_-^\perp$ so that

$$\Phi(x \otimes f^*) = U_1 x \otimes (V_1 f)^* = U_2 f \otimes (V_2 x)^*$$
$$\Phi(y \otimes f^*) = U_1 y \otimes (V_1 f)^* = U_2 f \otimes (V_2 y)^*$$

for arbitrary $x, y \in N$ and $f \in N_-^\perp$. It can be observed that $U_1 x \sim U_2 f$ and $U_2 f \sim U_1 y$, so $U_1 x \sim U_1 y$. Since the map U_1 can not have nonzero kernel, $x \sim y$. Using the same idea as this with $x \in N$ and $f, g \in N_-^\perp$, we can argue that $f \sim g$. These facts imply that $\dim N = \dim N_-^\perp = 1$. Observe that if $x \in N$, $x \otimes x^*$ is the

projection on N since $\dim N = 1$, so $x \otimes x^* \in \text{Alg}\,\mathfrak{L}$. Therefore, $x \in N_-^\perp$. Then $N = N_-^\perp$. Thus N is isolated. $\qquad\square$

LEMMA 2.9. *Let M and N be two subspaces in $\mathfrak{L}$ with $N \leq M$. Let Φ be a rank preserving map on $\text{Alg}\,\mathfrak{L}$. Then the followings are true.*

(1) *If M is consistent with respect to Φ, then N is also consistent with respect to Φ.*

(2) *If M is twisted with respect to Φ, then N is also twisted with respect to Φ.*

PROOF. First, let M be a consistent element in $\mathfrak{L}$ with respect to Φ. Without loss of generality, we assume $\dim N \geq 2$. Suppose that N is twisted with respect to Φ. Let U_M, V_M and U_N, V_N denote the two maps in lemma 2.5 defined on M and N ,respectively. By the assumption, we can choose two linearly independent vectors $x, y \in N$ and a vector $f \in M_-^\perp$. Note that $N_-^\perp \geq M_-^\perp$ since $N \leq M$. Since M is consistent with respect to Φ and N is twisted with respect to Φ, we have

$$\Phi(x \otimes f^*) = U_M x \otimes (V_M f)^* = U_N f \otimes (V_N x)^*$$

$$\Phi(y \otimes f^*) = U_M y \otimes (V_M f)^* = U_N f \otimes (V_N y)^*.$$

Then there exist two complex numbers λ and γ such that $U_M x = \lambda U_N f$ and $U_M y = \gamma U_N f$. But this contradicts the fact that $U_M x$ and $U_M y$ are linearly independent by Lemma 2.1, so the result (1) follows. The proof of (2) is essentially the same as the proof of (1). $\qquad\square$

LEMMA 2.10. *Let Φ be a rank preserving map on $\text{Alg}\,\mathfrak{L}$. Let M and N be two non-zero, non-isolated subspaces in $\mathfrak{L}$ with $M_- \neq I$. If M is consistent with respect to Φ, and N is twisted with respect to Φ, then*

$$M \wedge N = 0$$

and

$$M_-^\perp \wedge N_-^\perp = 0.$$

PROOF. Suppose $M \wedge N \neq 0$. Then $M \wedge N$ is a non-zero subspace of M and N. By Lemma 2.8, $M \wedge N$ is isolated, so $M \wedge N = (M \wedge N)_-^\perp = \langle e \rangle$ for some $e \in M \wedge N$. By Lemma 1.4, we have $(M \wedge N)_- \leq M_- \wedge N_-$. It follows that $(M \wedge N)_-^\perp \geq M_-^\perp \vee N_-^\perp$. Thus $M_-^\perp = N_-^\perp = (M \wedge N)_-^\perp = \langle e \rangle$. On the other hand, we have $M_- \geq N$ since $N \not\geq M$, so $\langle e \rangle = M_-^\perp \leq N^\perp$. Therefore $N \wedge N^\perp \geq \langle e \rangle \neq 0$ which gives a contradiction. Thus

$$M \wedge N = 0.$$

Observe that $M \vee N$ is neither consistent nor twisted with respect to Φ by Lemma 2.9. This implies $(M \vee N)_- = I$ by Lemma 2.5. Therefore

$$M_-^\perp \wedge N_-^\perp = (M_- \vee N_-)^\perp = (M \vee N)_-^\perp = 0.$$

$\qquad\square$

THEOREM 2.11. *Let $\mathfrak{L}$ be a completely distributive commutative subspace lattice and Φ be a rank preserving map on $\text{Alg}\,\mathfrak{L}$. Let*

$$M_i = \vee\{E \in \mathfrak{L} \,|\, E \text{ is isolated}\},$$

$$M_c = \vee\{E \in \mathfrak{L} \,|\, E \text{ is consistent with respect to}\, \Phi\} \ominus M_i$$

and

$$M_t = \vee\{E \in \mathfrak{L} \,|\, E \text{ is twisted with respect to}\, \Phi\ \} \ominus M_i.$$

Then

$$\mathrm{Alg}\,\mathfrak{L} = \mathrm{Alg}(M_i\mathfrak{L}M_i) \oplus \mathrm{Alg}(M_c\mathfrak{L}M_c) \oplus \mathrm{Alg}(M_t\mathfrak{L}M_t).$$

PROOF. By Lemma 2.10, it is clear that $M_i \oplus M_c \oplus M_t = I$. Since $M_i^\perp \in \mathfrak{L}$, $M_c, M_t \in \mathfrak{L}$. Note that $M_c^\perp = M_i \vee M_t$. Hence $M_c^\perp \in \mathfrak{L}$. Likewise, $M_t^\perp \in \mathfrak{L}$. The result follows. □

Remember that the operators U and V were constructed with the subspace N fixed. We now refer to them as U_N and V_N and try to fit together the U_N's and V_N's into operators U and V defined on the whole space $\mathcal{H}$ such that

$$\Phi(x \otimes f^*) = Ux \otimes (Vf)^*$$

whenever $x \otimes f^* \in \mathrm{Alg}\,\mathfrak{L}$. Note that if $M_- \neq I$, if $M < N$, and if $x \in M$ and $f \in N_-^\perp$, then $x \in N$ and $\Phi(x \otimes f^*) = (U_N x) \otimes (V_N f)^*$. On the other hand, $f \in N_-^\perp < M_-^\perp$, so $\Phi(x \otimes f^*) = (U_M x) \otimes (V_M f)^*$. Thus there exists a complex number λ such that $U_M x = \lambda U_N x$ and $V_N f = \overline{\lambda} V_M f$. Since x and f may vary independently, λ does not depend on x and f but only on M and N. We call it λ_{MN} and have

$$U_M = \lambda_{MN} U_N | M \quad \text{and} \quad V_N = \overline{\lambda_{MN}} V_M | N_-^\perp.$$

Let $\mathfrak{F}$ be the collection of all subspaces N in $\mathfrak{L}$ such that $N \neq 0$ and $N_- \neq I$.

REMARK 2.12. In the remainder of this paper we assume that every subspace in $\mathfrak{F}$ is consistent with respect to Φ.

Suppose that M and N lie in $\mathfrak{F}$. We will say that M and N are *comparable* if either $M \leq N$ or $N \leq M$. Suppose M and N are comparable. If $M \leq N$, λ_{MN} has already been defined. If $N \leq M$, define

$$\lambda_{MN} = \frac{1}{\lambda_{NM}} .$$

Thus λ_{MN} is defined whenever M and N are comparable, and it is easy to check that $\lambda_{LN} = \lambda_{LM} \lambda_{MN}$ whenever each pair from $\{L, M, N\}$ is comparable.

DEFINITION 2.13. Let M and N be two subspaces in $\mathfrak{F}$. We define a *chain* from M to N to be a finite sequence of subspaces $\{M_0, M_1, ..., M_n\}$, each M_k in $\mathfrak{F}$, such that $M_0 = M, M_n = N$, and such that M_k is comparable to M_{k+1} for each $k = 0, 1, ..., n - 1$.

If $M = N$, the chain $\{M_0, M_1, ..., M_n\}$ is called a *cycle of length* n.

Suppose $M, N \in \mathfrak{F}$. If there is a chain $\{M_0, M_1, ..., M_n\}$ from M to N, we want to define λ_{MN} to be $\lambda_{MM_1} \lambda_{M_1 M_2} \cdots \lambda_{M_{n-2} M_{n-1}} \lambda_{M_{n-1} N}$. Since there may be more than one chain from M to N, we need to show that such a product is well defined. The following lemma will prove this. An elaborate proof of the lemma can be found in [**3**].

LEMMA 2.14. *Let* $\{M_0, M_1, ..., M_n\}$ *be a cycle in* $\mathfrak{F}$. *Then*

$$\lambda_{M_0 M_1} \lambda_{M_1 M_2} \cdots \lambda_{M_{n-2} M_{n-1}} \lambda_{M_{n-1} M_n} = 1.$$

By making use of the Lemma 2.14, we can now show that

$$\lambda_{MN} = \lambda_{MM_1} \lambda_{M_1 M_2} \cdots \lambda_{M_{n-2} M_{n-1}} \lambda_{M_{n-1} N}$$

is well defined if there is a chain $\{M_0, M_1, ..., M_n\}$ from M to N.

Suppose that there are two chains $\{M_0, M_1, M_2, ..., M_{n-2}, M_{n-1}, M_n\}$ and $\{M_0, M_1', M_2', ..., M_{n-2}', M_{n-1}', M_n\}$ from M to N. We now form a cycle

$$\{M, M_1, ..., M_{n-1}, N, M_{n-1}', ..., M_1', M\}.$$

By applying Lemma 2.14 to this cycle, it follows that

$$\lambda_{MM_1} \cdots \lambda_{M_{n-1}N} \lambda_{NM_{n-1}'} \cdots \lambda_{M_1'M} = 1.$$

From this equation we get,

$$\lambda_{MM_1} \cdots \lambda_{M_{n-1}N} = \frac{1}{\lambda_{NM_{n-1}'} \cdots \lambda_{M_1'M}} = \lambda_{MM_1'} \cdots \lambda_{M_{n-1}'N}.$$

This shows that λ_{MN} is well defined.

Recall that $\mathfrak{F}$ denotes the collection of all N in $\mathfrak{L}$ such that $N \neq 0$ and $N_- \neq I$. Fix $N \in \mathfrak{L}$ and let

$$\mathfrak{G}_N^n = \{M \in \mathfrak{F} \,|\, M \text{ can be connected to } N \text{ by a chain of length } k \leq n\}.$$

Let $\mathfrak{G}_N = \cup_n \mathfrak{G}_N^n$.

LEMMA 2.15. *Let* $N \in \mathfrak{F}$. *Then there exist linear transformations* U *with dense domain in* $\vee\{M \,|\, M \in \mathfrak{G}_N\}$ *and* V *with dense domain in* $\vee\{M_-^\perp \,|\, M \in \mathfrak{G}_N\}$ *such that* $\Phi(x \otimes f^*) = Ux \otimes (Vf)^*$ *whenever there is* $M \in \mathfrak{G}_N$ *for which* $x \in M$ *and* $f \in M_-^\perp$.

PROOF. For $M \in \mathfrak{G}_N$, we have associated operators U_M and V_M such that

$$\Phi(x \otimes f^*) = U_M x \otimes (V_M f)^*,$$

whenever $x \in M$ and $f \in M_-^\perp$. Let $\widetilde{U}_M = \lambda_{NM} U_M$ and $\widetilde{V}_M = \overline{\lambda}_{MN} V_M$. Since $\lambda_{MN} \lambda_{NM} = 1$, we have $\Phi(x \otimes f^*) = \widetilde{U}_M x \otimes (\widetilde{V}_M f)^*$ for $x \in M$ and $f \in M_-^\perp$. Note that for $L, M \in \mathfrak{G}_N$ with $L \leq M$, if $\{N, N_1, ..., N_{n-1}, M\}$ is a chain from N to M, then $\{N, N_1, ..., N_{n-1}, M, L\}$ is a chain from N to L. Therefore $\lambda_{NL} = \lambda_{NM} \lambda_{ML}$. Thus, if $x \in L$, we have $\widetilde{U}_L x = \lambda_{NL} U_L x = \lambda_{NM} \lambda_{ML} U_L x$. On the other hand, by definition of λ_{LM} we have $U_L x = \lambda_{LM} U_M x$, so $\widetilde{U}_L x = \lambda_{NM} \lambda_{ML} \lambda_{LM} U_M x = \lambda_{NM} U_M x = \widetilde{U}_M x$. Thus, if $L \leq M$, $\widetilde{U}_L$ and $\widetilde{U}_M$ agree on L. Let $\mathcal{M} = \{x_1 + \cdots + x_n \,|\, \text{for some positive integer } n, x_i \in M \text{ for some } M \in \mathfrak{G}_N\}$. Define a linear transformation U on $\mathcal{M}$ by $U|M = \widetilde{U}_M$. By the coherence of the $\widetilde{U}_M$, the map U is well defined. Similarly, we can define $V|M_-^\perp = \widetilde{V}_M$. If $x \in M$, $f \in M_-^\perp$ for $M \in \mathfrak{G}_N$, then we have $\Phi(x \otimes f^*) = Ux \otimes (Vf)^*$. $\square$

LEMMA 2.16. *For each* $N \in \mathfrak{F}$, *let* $G_N = \vee\{M \,|\, M \in \mathfrak{G}_N\}$ *and* $F_N = \vee\{M_-^\perp \,|\, M \in \mathfrak{G}_N\}$. *Let* $N, N' \in \mathfrak{F}$.

    (1) *If* $N' \in \mathfrak{G}_N$, *then* $G_N = G_{N'}$ *and* $F_N = F_{N'}$
    (2) *If* $N' \notin \mathfrak{G}_N$, *then* $G_N \perp G_{N'}$ *and* $F_N \perp F_{N'}$

PROOF. Suppose that $N' \in \mathfrak{G}_N$. Note that if there exists a chain connecting N to N' and one connecting N' to M, then there is a chain connecting N to M, whence $\mathfrak{G}_N = \mathfrak{G}_{N'}$. Thus $G_N = G_{N'}$ and $F_N = F_{N'}$. Suppose that $N' \notin \mathfrak{G}_N$; then, for any $M' \in \mathfrak{G}_{N'}$, $M' \notin \mathfrak{G}_N$. For such an M', consider the projection NM'. If $NM' \neq 0$, then $NM' \in \mathfrak{G}_N \cap \mathfrak{G}_{M'} = \mathfrak{G}_{M'} \cap \mathfrak{G}_N$. Since this is impossible, it must be that $NM' = 0$. Likewise, if $M \in \mathfrak{G}_N$, $MM' = 0$ and hence $G_N G_{N'} = 0$. $\square$

THEOREM 2.17. *Let $\mathfrak{L}$ be a completely distributive commutative subspace lattice and let Φ be a weakly continuous rank preserving map defined on $\mathrm{Alg}\,\mathfrak{L}$. If we assume that $\mathfrak{F} = \mathfrak{G}_N$ for some $N \in \mathfrak{F}$ (i.e. $\mathrm{Alg}\,\mathfrak{L}$ is irreducible.), then there exist two densely defined linear transformations U, V such that*

$$\Phi(A) = UAV^*$$

for all $A \in \mathrm{Alg}\,\mathfrak{L}$.

PROOF. By Lemma 2.15, there exist densely defined maps U and V such that $\Phi(x \otimes f^*) = Ux \otimes (Vf)^*$ whenever $x \otimes f^* \in \mathrm{Alg}\,\mathfrak{L}$. Since $\mathfrak{F} = \mathfrak{G}_N$ for some $N \in \mathfrak{F}$, the domain of U is the nonclosed linear span of $\mathfrak{F} = \{E \in \mathfrak{L} \,|\, E_- \neq I\}$ and the domain of V is the nonclosed linear span of $\{M_-^\perp \,|\, M \in \mathfrak{F}\}$. By Lemma 1.3, these domains are dense in $\mathcal{H}$. Recall that $R_\mathfrak{L}$ denote the linear span of the rank one operators in $\mathrm{Alg}\,\mathfrak{L}$. Therefore, for any operator $A \in R_\mathfrak{L}$, $\Phi(A) = UAV^*$ by the linearity of Φ. By Lemma 1.1, in CDC algebra it is guaranteed that there exists a net of operators in $R_\mathfrak{L}$ weakly converging to any operator in $\mathrm{Alg}\,\mathfrak{L}$. Let A be an operator in $\mathrm{Alg}\,\mathfrak{L}$. Then there exists a net $\{A_\alpha\}_{\alpha \in J}$ such that A_α converges weakly to A. For each $h \in \mathrm{dom}\,V^*$ and $k \in \mathrm{dom}\,U^*$,

$$\begin{aligned}
\langle UA_\alpha V^* h, k \rangle &= \langle A_\alpha V^* h, U^* k \rangle \\
&\to \langle AV^* h, U^* k \rangle \text{ since } A_\alpha \text{ weakly converges to } A \\
&= \langle UAV^* h, k \rangle.
\end{aligned}$$

Thus

$$\Phi(A_\alpha) = UA_\alpha V^* \xrightarrow{\;w\;} UAV^*.$$

On the other hand, $\Phi(A_\alpha)$ converges weakly to $\Phi(A)$ since Φ is weakly continuous. Thus $\Phi(A) = UAV^*$ for $A \in \mathrm{Alg}\,\mathfrak{L}$.

$\square$

Now we are finally in a position to state and prove the main result of this paper.

THEOREM 2.18. *Let $\mathfrak{L}$ be a completely distributive commutative subspace lattice and let Φ be a weakly continuous rank preserving map defined on $\mathrm{Alg}\,\mathfrak{L}$. If we assume that every subspace is consistent with respect to Φ, then there exist two densely defined linear transformations U, V such that*

$$\Phi(A) = UAV^*$$

for all $A \in \mathrm{Alg}\,\mathfrak{L}$.

PROOF. Let $N \in \mathfrak{F}$ and consider the collection $\mathfrak{G}_N$. If $\mathfrak{G}_N = \mathfrak{F}$, then this is nothing but the case of Proposition 2.17. Suppose there exists $M \in \mathfrak{F}$ with $M \notin \mathfrak{G}_N$ such that $G_N G_M = G_M G_N = 0$. In this way we can form a sequence $\{G_{N_i}\}$ of mutually orthogonal projections in $\mathfrak{L}$. We shall suppress the N and write simply G_i. The separability of $\mathcal{H}$ guarantees that there are no more than countably many G_i's. We have $\vee G_i = I$ since $\vee\{N \,:\, N \in \mathfrak{F}\} = I$, whence $G_j^\perp = \vee_{i \neq j} G_i$ is also in $\mathfrak{L}$. Therefore, the algebra $\mathrm{Alg}\,\mathfrak{L}$ can be written as the direct sum $\sum_i \oplus \mathrm{Alg}(G_i \mathfrak{L} G_i)$. For any $A \in \mathrm{Alg}\,\mathfrak{L}$, we can write $A = \sum_i \oplus A_i$ where $A_i \in \mathrm{Alg}(G_i \mathfrak{L} G_i)$. By Proposition 2.17, there exist two densely defined linear maps U_i and V_i such that $\Phi(A_i) = U_i A_i V_i^*$ for each i. If we define $U = \sum_i \oplus U_i$ and $V = \sum_i \oplus V_i$, then U and V are densely defined on $\mathcal{H}$. Then the result follows. $\square$

Corollary. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\mathfrak{L}$ be a commutative subspace lattice consisting of subspaces in $\mathcal{H}$. Let Φ be a rank preserving map on $\mathrm{Alg}\,\mathfrak{L}$. If we assume that every element in $\mathfrak{L}$ is consistent with respect to Φ, then there exists a map Ψ defined by $\Psi(A) = A\Phi^{-1}(I)$ such that $\Psi \circ \Phi$ is an isomorphism on $\mathrm{Alg}\,\mathfrak{L}$.*

PROOF. By Main Theorem, there are two linear maps U and V such that $\Phi(A) = UAV^*$. Since $\mathcal{H}$ is finite dimensional, it is true that $\Phi(I) = UV^*$ is invertible. Therefore $\Psi(A) = A\Phi^{-1}(I) = A(UV^*)^{-1}$ is a well defined map. Then

$$(\Psi \circ \Phi)(A) = \Phi(A)(V^*)^{-1}U^{-1}$$

$$= UAV^*(V^*)^{-1}U^{-1}$$
$$= UAU^{-1}$$

Thus $\Psi \circ \Phi$ is an isomorphism implemented by U. ☐

3. EXAMPLES

In this section we present some examples of rank preserving maps. Most of the examples we discuss in this section will be either $\mathcal{A}_{2n}$ or $\mathcal{A}_{\infty}$. The precise definition of these algebras is given in [**2**]. But for our discussion it is enough to say that the algebras $\mathcal{A}_{2n}$ are tridiagonal matrices, of size $2n \times 2n$, of the form

$$
\begin{bmatrix}
* & * & & & & & & & & * \\
 & * & & & & & & & & \\
 & * & * & * & & & & & & \\
 & & & * & & & & & & \\
 & & & * & * & * & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & & * & & \\
 & & & & & & & * & & \\
 & & & & & & & * & * & * \\
 & & & & & & & & & *
\end{bmatrix}
$$

where all nonstarred entries are 0. It can be observed that the associated lattice consists of certain diagonal projections, hence it is commutative and completely distributive. Thus the algebra $\mathcal{A}_{2n}$ is reflexive. The algebra $\mathcal{A}_{\infty}$ consists of infinite matrices of the form

$$
\begin{bmatrix}
* & * & & & \\
 & * & & & \\
 & * & * & * & \\
 & & & * & \\
 & & & * & \\
 & & & & \ddots
\end{bmatrix}.
$$

Once again the associated lattice is commutative and completely distributive, so the algebra $\mathcal{A}_{\infty}$ is reflexive.

EXAMPLE 3.1. Consider $\mathcal{A}_4$. Define a map $\Phi_1 : \mathcal{A}_4 \to \mathcal{A}_4$ by

$$
\begin{bmatrix} a & b & 0 & h \\ 0 & c & 0 & 0 \\ 0 & d & e & f \\ 0 & 0 & 0 & g \end{bmatrix} \mapsto \begin{bmatrix} e & f & 0 & d \\ 0 & g & 0 & 0 \\ 0 & h & a & b \\ 0 & 0 & 0 & c \end{bmatrix}.
$$

Then it is easy to check that Φ_1 is rank preserving. Moreover, the map Φ_1 is implemented by

$$
U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
$$

In other words, the map Φ_1 is an isomorphism such that

$$
\Phi_1(A) = UAU^{-1}.
$$

If we define a map $\Phi_2 : \mathcal{A}_4 \to \mathcal{A}_4$ by

$$
\begin{bmatrix} a & b & 0 & h \\ 0 & c & 0 & 0 \\ 0 & d & e & f \\ 0 & 0 & 0 & g \end{bmatrix} \mapsto \begin{bmatrix} g & f & 0 & h \\ 0 & e & 0 & 0 \\ 0 & d & c & b \\ 0 & 0 & 0 & a \end{bmatrix},
$$

then Φ_2 is also a rank preserving map. But the map Φ_2 is not an isomorphism. Instead, we can write the map Φ_2 as

$$
\Phi_2(A) = VA^T V^{-1}
$$

where

$$
V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
$$

EXAMPLE 3.2. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B} = \{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let $\mathcal{N} = \{0, [e_1], [e_1, e_2], [e_1, e_2, e_3], ...\}$. Then $\mathcal{N}$ is a nest and the corresponding nest algebra is

$$
\text{Alg}\,\mathcal{N} = \begin{bmatrix} * & * & * & * & * & \cdots \\ 0 & * & * & * & * & \cdots \\ 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & * & * & \cdots \\ 0 & 0 & 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
$$

Let S denote the unilateral shift operator such that $Se_k = e_{k+1}$. If we define $\Phi : \text{Alg}\,\mathcal{N} \longrightarrow \text{Alg}\,\mathcal{N}$ by $\Phi(A) = SAS^*$, then

$$
\Phi : \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\ 0 & 0 & a_{33} & a_{34} & a_{35} & \cdots \\ 0 & 0 & 0 & a_{44} & a_{45} & \cdots \\ 0 & 0 & 0 & 0 & a_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ 0 & 0 & a_{22} & a_{23} & a_{24} & \cdots \\ 0 & 0 & 0 & a_{33} & a_{34} & \cdots \\ 0 & 0 & 0 & 0 & a_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
$$

From this fact we can observe that Φ is a rank preserving map on $\mathrm{Alg}\,\mathcal{N}$ but it is not an onto map.

EXAMPLE 3.3. Consider the algebra $\mathcal{A}_\infty$ which is described at the beginning of Section 3. Let $\mathfrak{L}$ be the lattice associated with $\mathcal{A}_\infty$. We can easily argue that for any $M, N \in \mathfrak{L}$, $\mathfrak{G}_M = \mathfrak{G}_N$. Therefore $\mathcal{A}_\infty$ is irreducible. Let Φ be a map on $\mathcal{A}_\infty$ defined by

$$\Phi : \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & \cdots \\ 0 & c & 0 & 0 & 0 & 0 \\ 0 & d & e & f & 0 & 0 \\ 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & i & j \\ \vdots & & & & & & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} a & \frac{1}{2}b & 0 & 0 & 0 & 0 & \cdots \\ 0 & c & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}d & e & \frac{3}{4}f & 0 & 0 \\ 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4}h & i & \frac{5}{6}j \\ \vdots & & & & & & \ddots \end{bmatrix}.$$

Then it is obvious that the map $\Phi : \mathcal{A}_\infty \to \mathcal{A}_\infty$ preserves rank and it can also be observed that the map Φ is implemented by an unbounded operator

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ \vdots & & & & & \ddots \end{bmatrix}.$$

In other words,

$$\Phi(A) = UAU^{-1}$$

for all $A \in \mathrm{Alg}\,\mathfrak{L}$.

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